PMM U.S.S.R., Vol.54, No.3, pp.421-424, 1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

ON A METHOD OF SOLVING A SHAPE OPTIMIZATION PROBLEM IN ELASTICITY THEORY*

I.N. KANDOBA

The problem of calculation of the shape of a doubly-connected transverse section of an elastic homogeneous prismatic rod possessing maximum torsional stiffness is considered. The shape of its internal prismatic cavity is here assumed to be fixed while the area of the rod transverse cross-section is constant.

Certain special properties are set up for solving the Dirichlet problem for the equation $\Delta U = 1$ in a bounded closed domain of two-dimensional Euclidean space.

Application of the results obtained in the construction of a numerical method of solving a number of optimization domain problems for elliptical systems is illustrated by an example of the problem under consideration. The paper touches on the investigations in /1, 2/.

We consider a plane doubly-connected domain $\Omega(\Gamma^*, \Gamma)$ of the two-dimensional Euclidean space R^2 bounded by smooth non-intersecting Jordan curves $\Gamma^* \in C^1$ and $\Gamma \in C^2$ (Fig.1).

Let $U(\Gamma^*, \Gamma; p)$ denote the solution of the boundary-value problem describing the torsion state /3/

$$-\Delta U(\Gamma^*, \Gamma; p) = 1,\ p \in \Omega(\Gamma^*, \Gamma);\ U(\Gamma^*, \Gamma; p) = 0,\ p \in \Gamma \tag{1}$$
$$U(\Gamma^*, \Gamma; p) = \text{const}\ (p \in \Gamma^*):\ \int_{\Gamma^*} D_n U(\Gamma^*, \Gamma; p)\,|dp| = \operatorname{mes}(\Omega^*)$$

where D_n is the derivative with respect to the direction of the external normal to the contour Γ^* bounding the domain Ω^*, $\operatorname{mes}(\Omega^*)$ is the Lebesgue measure of the domain Ω^*.

We shall assume that $\Omega(\Gamma^*, \Gamma)$ is the transverse cross-section of the rod. Then the function $U(\Gamma^*, \Gamma; p)$ yields the stress distribution in this section that occurs under torsion of this rod. The torsional stiffness of the rod is here determined by the value of the functional

$$J(U(\Gamma^*, \Gamma; p)) = \int_{\Omega(\Gamma^*, \Gamma)} \varphi^2(p)\,dp$$
$$\varphi(p) = |\nabla U(\Gamma^*, \Gamma; p)|,\ p \in \Omega(\Gamma^*, \Gamma)$$

We will introduce the notation

$$c(\Gamma) = \max\{U(\Gamma^*, \Gamma; p) \mid p \in \Omega(\Gamma^*, \Gamma)\},\ I(\Gamma) = (0, c(\Gamma))$$
$$\Gamma_c = \{p \in \Omega(\Gamma^*, \Gamma) \mid U(\Gamma^*, \Gamma; p) = c,\ c \in I(\Gamma)\}$$
$$B(\Gamma) = \{\Gamma_c \mid c \in I(\Gamma)\},\ \varepsilon(\Gamma) = 1/\varkappa(\Gamma)$$
$$\varkappa(\Gamma) = \max\{|\varkappa(\Gamma, p)| \mid p \in \Gamma\}$$

where $\varkappa(\Gamma, p)$ is the curvature of the contour Γ at the point p. By virtue of the assumptions made about the smoothness and closeness of the contour Γ the function $\varkappa(\Gamma)$ is positive and bounded /4/.

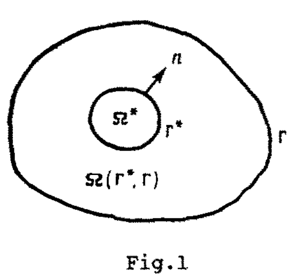

Fig.1

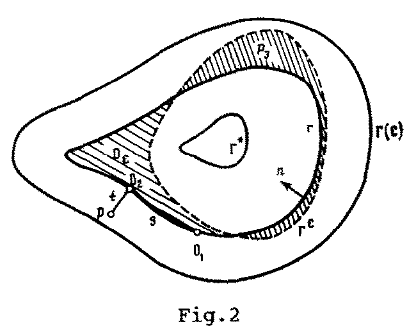

Fig.2

*Prikl. Matem. Mekhan., 54, 3, 506-510, 1990

We shall understand an ε-neighbourhood of the domain $\Omega(\Gamma^*, \Gamma)$ to be the domain

$$\Omega(\Gamma^*, \Gamma(\varepsilon)) = \bigcup_p \{B(p, \varepsilon) \setminus \Omega^* \mid p \in \Omega(\Gamma^*, \Gamma)\}, \ \varepsilon > 0$$
$$B(p, \varepsilon) = \{q \in R^2 \mid | p - q | < \varepsilon\}$$

where $\Gamma(\varepsilon)$ is the external component of the domain boundary

$$\Omega(\Gamma^*, \Gamma(\varepsilon)).$$

A numerical iteration method /2/ was proposed for calculating the desired section shape, where the inequality (Theorem 1 in /2/)

$$J(U(\Gamma^*, \Gamma^\varepsilon; p)) - J(U(\Gamma^*, \Gamma; p)) \geqslant \Phi(\Gamma^*, \Gamma; \varepsilon) \tag{2}$$
$$\Phi(\Gamma^*, \Gamma; \varepsilon) = \int_{\Omega^\varepsilon} \varphi^2(\varepsilon, p)\, dp - \int_{\Omega} \varphi^2(\varepsilon, p)\, dp, \quad \varepsilon \in (0, \delta(\Gamma))$$
$$\varphi(\varepsilon, p) = | \nabla U(\Gamma^*, \Gamma(\varepsilon); p) |, \ p \in \Omega(\Gamma^*, \Gamma(\varepsilon))$$
$$\Omega = \Omega(\Gamma^*, \Gamma), \ \Omega^\varepsilon = \Omega(\Gamma^*, \Gamma^\varepsilon) \subseteq \Omega(\Gamma^*, \Gamma(\varepsilon)): \Gamma^\varepsilon \in B(\Gamma(\varepsilon))$$
$$\operatorname{mes}(\Omega^\varepsilon) = \operatorname{mes}(\Omega) = S; \ \delta(\Gamma) = \max\{0 < \delta \leqslant \varepsilon(\Gamma) \mid \Gamma^\delta\}$$

is the basis of the method of Γ^δ is a simply-connected contour. We note with respect to $\delta(\Gamma)$ that by virtue of the known /5/ necessary optimality conditions in the problem under consideration here, it is natural to limit oneself to a class of domains $\Omega(\Gamma^*, \Gamma)$, on whose boundary Γ the absolute value of the gradient of the solution of (1) is uniformly separated from zero. And for such domains $\delta(\Gamma) > 0$.

A domain bounded by an inner contour Γ^* and an external Γ^ε where $\varepsilon \in (0, \delta(\Gamma))$ (Fig.2), is taken as the next approximation of the desired section form at each step of this method. In order for an improvement in the quality functional to occur here, it is obviously sufficient that a $\varepsilon^* \in (0, \delta(\Gamma)]$, exist for which the inequality $\Phi(\Gamma^*, \Gamma; \varepsilon^*) > K\varepsilon^*$, should be satisfied, where K is a certain positive constant.

The following theorem yields a qualitative interpretation to this last inequality in terms, substantially, of the necessary optimality conditions.

Theorem 1. Let $\Omega(\Gamma^*, \Gamma)$ be a certain doubly-connected closed domain from R^2 that possesses the above-mentioned properties. Then for sufficiently small $\varepsilon > 0$ the following inequality holds:

$$\Phi(\Gamma^*, \Gamma; \varepsilon) - \Phi(\Gamma^*, \Gamma; 0) \geqslant K(\Gamma)\, \varepsilon \geqslant 0 \tag{3}$$
$$K(\Gamma) = \int_\Gamma \varphi^2(p) \, | dp | - L^{-1} \Big(\int_\Gamma \varphi(p) \, | dp | \Big)^2$$

where $L = L(\Gamma)$ is the length of the contour Γ.

Before proving the theorem, we will prove an auxiliary assertion.

Proposition 1. Let the function $f(\varepsilon, x) \in C^{0,m}([0, \delta] \times [a, b])$ and, moreover $f_x^{(k)}(\varepsilon, x) \in C^{0,\, m-k}([0, \delta] \times [a, b])$ $(m \in N, \ k = 1, 2, \ldots, m-1)$. Then for any $\nu \in [0, \delta]$ as $\varepsilon \to \nu$ $f_x^{(k)}(\varepsilon, x) \to f_x^{(k)}(\nu, x)$ uniformly in x on $[a, b]$ as $\varepsilon \to \nu$ for all $k = 0, 1, 2, \ldots, m-1$.

Indeed for any $\nu \in [0, \delta]$ and all $k = 0, 1, 2, \ldots, m-1$ the following inequalities hold $(\varepsilon \in [0, \delta], x \in [a, b])$:

$$0 \leqslant | G_k(\varepsilon, x) | \leqslant \Big| \int_a^x G_{k+1}(\varepsilon, y)\, dy \Big| + | G_k(\varepsilon, a) | \leqslant F_k(\varepsilon) + | G_k(\varepsilon, a) |,$$
$$F_k(\varepsilon) = \int_a^b | G_{k+1}(\varepsilon, y) | \, dy, \quad G_k(\varepsilon, x) = f_x^{(k)}(\varepsilon, x) - f_x^{(k)}(\nu, x)$$

By virtue of the assumptions made about the smoothness of the function $f(\varepsilon, x)$ and the Lebesgue theorem, we have $F_k(\varepsilon) \to 0$ and $| G_k(\varepsilon, a) | \to 0$ as $\varepsilon \to \nu$ for all $k = 0, 1, 2, \ldots, m-1$. Hence the assertion to be proved indeed follows.

We will now prove the theorem. We introduce the sets $P(\varepsilon) = \overline{\Omega(\Gamma^*, \Gamma(\varepsilon)) \setminus \Omega(\Gamma^*, \Gamma)}$, $P_\varepsilon = \overline{P(\varepsilon) \setminus D(\varepsilon)}$, $D(\varepsilon) = \overline{\Omega(\Gamma^*, \Gamma(\varepsilon)) \setminus \Omega^\varepsilon}$, $P_\varepsilon = \overline{P(\varepsilon) \setminus D(\varepsilon)}$ (Fig.2) into the considerations. Using Green's formula, the validity of the equation

$$\Phi(\Gamma^*, \Gamma; \varepsilon) - \Phi(\Gamma^*, \Gamma; 0) = A + B - C$$
$$U(\varepsilon; p) = U(\Gamma^*, \Gamma(\varepsilon), p)$$

$$A = \int_{P_\varepsilon} U(\varepsilon;\ p)\, dp - \int_{D_\varepsilon} U(\varepsilon;\ p)\, dp, \quad B = \int_{\Gamma} U(\varepsilon;\ p)\, D_n U(\varepsilon;\ p) \,|\, dp\,|,$$

$$C = \int_{\Gamma^\varepsilon} U(\varepsilon;\ p)\, D_n U(\varepsilon;\ p) \,|\, dp\,|$$

can be seen.

Here D_n is the derivative with respect to the direction of the external normal with respect to $P(\varepsilon)$ (in the expression for B) and $D(\varepsilon)$ (in the expression for C) to the appropriate contour.

By the definition of Γ^ε we have $U(\varepsilon;\ p) = c(\varepsilon),\ p \in \Gamma^\varepsilon$, where $c(\varepsilon) > 0$ is a certain constant which depends only on $\Gamma^*, \Gamma, \varepsilon$ and S. By the definition of P_ε and D_ε, the inequalities $U(\varepsilon;\ p) \geqslant c(\varepsilon)$ and $U(\varepsilon;\ q) \leqslant c(\varepsilon)$, respectively, hold for any points $p \in P_\varepsilon$ and $q \in D_\varepsilon$. And since by definition $\Phi(\Gamma^*,\ \Gamma;\ \varepsilon)$, $\operatorname{mes}(P_\varepsilon) = \operatorname{mes}(D_\varepsilon)$, we have $A \geqslant c(\varepsilon) \operatorname{mes}(D_\varepsilon) \geqslant 0$. We hence obtain that

$$\Phi(\Gamma^*,\ \Gamma;\ \varepsilon) - \Phi(\Gamma^*,\ \Gamma;\ 0) \geqslant B - c(\varepsilon) \int_{\Gamma^\varepsilon} \varphi(\varepsilon,\ p) \,|\, dp\,| = B - c(\varepsilon) \int_{\Gamma} \varphi(p) \,|\, dp\,| \tag{4}$$

By the definition of $\delta(\Gamma)$ and $\Gamma(\varepsilon)$ a single point $p(\varepsilon) \in \Gamma(\varepsilon)$, can be set in correspondence with each point $p \in \Gamma$ such that the points p and $p(\varepsilon)$ lie on the segment $T(p,\ p(\varepsilon))$ connecting them and perpendicular to the contours Γ and $\Gamma(\varepsilon)$ at the points corresponding to them.

We have from the formula of finite increments

$$U(\varepsilon;\ p) = (\nabla U(\varepsilon;\ p_0),\ p - p(\varepsilon))\, \varepsilon,\ p_0 \in T(p,\ p(\varepsilon))$$

whence by virtue of the continuity of $U(\varepsilon;\ p)$ in $\Omega(\Gamma^*, \Gamma(\varepsilon))$ it follows that $U(\varepsilon;\ p) = (D_n U(\varepsilon;\ p) + \psi(\varepsilon,\ p))\, \varepsilon$, where $\psi(\varepsilon,\ p)$ is a certain function continuous in p on Γ, where $\psi(\varepsilon,\ p) \to 0$ uniformly in $p \in \Gamma$ as $\varepsilon \to 0$. Therefore

$$B = \varepsilon \int_{\Gamma} ((D_n U(\varepsilon;\ p))^2 + \sigma(\varepsilon,\ p)) \,|\, dp\,| \tag{5}$$

$$\sigma(\varepsilon,\ p) = \psi(\varepsilon,\ p)\, D_n U(\varepsilon;\ p)$$

For convenience we introduce a new $s,\ t$ coordinate system associated with the reference line Γ (Fig.2). The coordinate of the point $p \in P(\varepsilon) \cup D(\varepsilon)$ is measured along Γ from a certain fixed point $O_1 \in \Gamma$ to the point O_2 of intersection of Γ with the internal normal to Γ with respect to $\Omega(\Gamma^*, \Gamma)$ that passes through the point p. The coordinate t equals the length of the segment $O_2 p$ taken with a sign which depends on whether the point p belongs to the domain $\Omega(\Gamma^*, \Gamma)$ (the plus sign) or not (the minus sign).

Then for sufficiently small $\varepsilon > 0$ the contour Γ^ε in $s,\ t$ coordinates can be given by the equation $t = \rho(\varepsilon,\ s),\ s \in [0,\ L(\Gamma)]$. For each $\varepsilon > 0$ the direction of traversal over the reference contour Γ from the point O_1 to the point O_2 is here selected such that

$$\int_{\Gamma} \varkappa(\Gamma,\ p(s,\ 0))\, \rho^2(\varepsilon,\ p(s,\ 0)) \,|\, dp\,| =$$

$$\int \varkappa(\Gamma,\ s)\, \rho^2(\varepsilon,\ s)\, ds \geqslant 0,\ \varkappa(\Gamma,\ s) = \varkappa(\Gamma,\ p(s,\ 0))$$

Here and henceforth the itegration with respect to s is between 0 and $L = L(\Gamma)$.

Consider the expression

$$G(\varepsilon) = L^{-1} \int (U(\varepsilon;\ p(s,\ \rho(\varepsilon,\ s))) - U(\varepsilon;\ p(s,\ -\varepsilon)))\, ds$$

Obviously $G(\varepsilon) = c(\varepsilon)$. On the other hand, by applying the finite increment formula to the expression under the integral sign, we obtain

$$c(\varepsilon) = L^{-1} \int F(\varepsilon,\ s)\, (\varepsilon + \rho(\varepsilon,\ s))\, ds$$

$$F(\varepsilon,\ s) = D_n U(\varepsilon;\ p(s,\ 0)) + \nu(\varepsilon,\ p(s,\ 0))$$

where $\nu(\varepsilon,\ p(s,\ 0)) \to 0$ uniformly on Γ as $\varepsilon \to 0$. The estimates

$$F(\varepsilon,\ s) < \varepsilon^{-1} c(\varepsilon),\ \forall s \in [0,\ L(\Gamma)]\colon \rho(\varepsilon,\ s) > 0$$

$$F(\varepsilon,\ s) > \varepsilon^{-1} c(\varepsilon),\ \forall s \in [0,\ L(\Gamma)]\colon \rho(\varepsilon,\ s) < 0$$

hold here.

Therefore

$$\int F(\varepsilon, s)\,\rho(\varepsilon, s)\,ds < \varepsilon^{-1}c(\varepsilon)\int \rho(\varepsilon, s)\,ds <$$
$$\varepsilon^{-1}c(\varepsilon)\int \left(\rho(\varepsilon, s) + \frac{1}{2}\varkappa(\Gamma, s)\,\rho^2(\varepsilon, s)\right) ds =$$
$$\varepsilon^{-1}c(\varepsilon)\int \int\limits_0^{\rho(\varepsilon, s)} (1 + t\varkappa(\Gamma, s))\,dt\,ds = \varepsilon^{-1}c(\varepsilon)\,(\mathrm{mes}\,(D_\varepsilon) - \mathrm{mes}\,(P_\varepsilon)) = 0$$

Hence $c(\varepsilon) < L^{-1}\varepsilon\int F(\varepsilon, s)\,ds$, and taking account of the relationships (4) and (5) we obtain the estimate

$$\Phi(\Gamma^*, \Gamma;\, \varepsilon) - \Phi(\Gamma^*, \Gamma;\, 0) > \varepsilon\left(\int\limits_\Gamma (D_n U(\varepsilon;\, p))^2\,|dp| - L^{-1}\left(\int\limits_\Gamma D_n U(\varepsilon;\, p)\,|dp|\right)^2\right) + o(\varepsilon) \tag{6}$$

By virtue of Lemma 7 from /6/

$$U(\varepsilon;\, p(s, 0)) = U(\varepsilon, s) \in C^{0,1}([0, \delta(\Gamma)] \times [0, L(\Gamma)])$$
$$D_n U(\varepsilon, s) \in C^{0,0}([0, \delta(\Gamma)] \times [0, L(\Gamma)])$$

The inequality (3) follows from Proposition 1 and inequality (6). We obtain here from the Hölder inequality that $K(\Gamma) \geqslant 0$.

Proposition 2. Let $f(x) \in C^2[a, b]$. Then to satisfy the equality

$$\langle f^2\rangle = (b-a)^{-1}\langle f\rangle^2\left(\langle f\rangle = \int\limits_a^b f(x)\,dx\right) \tag{7}$$

it is necessary and sufficient that $f(x) \equiv \mathrm{const},\ x \in [a, b]$.

It can be seen that the difference between the left and right sides of (7) equals $\|f(x) - K\|_{L_2[a, b]}$, where $K = (b-a)^{-1}\langle f\rangle$ from which the assertion to be proved indeed follows.

In conclusion, we note certain applications of the results obtained. We assume that the form of the rod transverse cross-section occupies a domain $\Omega(\Gamma^*, \Gamma)$ and is not optimal, i.e., $\max\{\varphi(p) \mid p \in \Gamma\} - \min\{\varphi(p) \mid p \in \Gamma\} > 0$ /5/. Then it follows from inequality (2), Theorem 1 and Proposition 2 that a $\varepsilon > 0$ exists such that the rod whose transverse section occupies the domain $\Omega(\Gamma^*, \Gamma^\varepsilon)$ possesses a greater torsional stiffness than a rod with the initial form of transverse section. Having been given a certain initial domain and realizing the selection of such an ε each time, a maximizing sequence of domains can be constructed in the domain optimization problem under consideration here. It can be shown that a decrease of the residual of the absolute value of the gradient of the solution of the problem (1) occurs along this sequence on the boundary of each succeeding domain.

Analogous assertions (to the accuracy of the problem formulation) hold even for problems of minimizing the thermal flux through the wall of a prismatic tube on whose transverse cross-section an isoperimetric constant is imposed.

The author is grateful to YU.S. Osipov and A.P. Suetov for his interest and for useful remarks.

REFERENCES

1. ACKER A., Heat flow inequalities with applications to heat flow optimization problems, SIAM J. Math. Analysis, 8, 4, 1977.
2. KANDOBA I.N., On a form optimization algorithm in elliptical systems, PMM, 53, 2, 1989.
3. LUR'YE A.I., Theory of Elasticity, Nauka, Moscow, 1970.
4. BRUCE J. and GIBLIN P., Curves and Singularities, Mir, Moscow, 1988.
5. BANICHUK N.V., Optimization of the Form of Elastic Bodies. Nauka, Moscow, 1980.
6. KREIN G., Behaviour of solutions of elliptical problems during domain variation, Studia Math., 31, 4, 1968.

Translated by M.D.F.